Tannery's explanation of the four arguments, particularly of the "Arrow" and "Stade," raises these paradoxes from childish arguments to arguments with conclusions which follow with compelling force. It does not place Zeno in the position of being ignorant of the most simple ideas of relative motion; it exhibits Zeno as a logician of the first rank.

Tannery's conclusions have been strongly supported by G. Milhaud,[1] but opposed by other French writers and by Zeller.

CENTERS OF SIMILITUDE OF CIRCLES AND CERTAIN THEOREMS ATTRIBUTED TO MONGE. WERE THEY KNOWN TO THE GREEKS?

By RAYMOND CLARE ARCHIBALD, Brown University.

One of the most noticeable characteristics of French, German and Italian, as opposed to American, texts on elementary geometry is the emphasis laid on broad underlying principles. How many American high-school graduates could give one any idea of the theory of similitude of plane and solid figures? How many teachers realize the importance of this far reaching theory in the solution of geometrical problems, or are familiar with the equivalent of Petersen's excellent exposition?[2] At all events it seems well worth while to draw attention to some simple results in the theory, and to put on record their historical setting. The theorems attributed to Monge, which I propose to discuss, involve the centers of similitude of circles (spheres).

The centers of similitude of two circles (spheres), whose centers are A and B, are the points which divide AB internally, at C, and externally, at D, in the ratio of the radii, r_a, r_b $(r_a \geqq r_b)$;

$$AC : CB = AD : BD = r_a : r_b.$$

The circles (spheres) may be situated in any fashion. If they are tangent (internally or externally), the point of tangency is a center of similitude. If concentric, we may, perhaps, say that either A, B, C, D coincide or else A, B, C coincide while D is indeterminate. If they are equal and non-concentric, D is at infinity[3] and C bisects AB.

Lines joining the ends of parallel radii pass through a center of similitude, and common tangent lines (planes), when they exist, also pass through such a center. Conversely, if through a center of similitude, D (or C), of two circles a line be

[1] G. Milhaud, "Le concept du nombre chez les Pythagoriciens et les Éléates," *Revue de métaphysique et de morale*, I, Paris, 1893, p. 141.

[2] J. Petersen, *Methods and Theories for the Solution of Problems of Geometrical Constructions*, Copenhagen, 1879, pp. 22 ff. This is an English edition of the remarkable Danish original. There are also French, German, Italian, Hungarian and Russian translations making in all some 15 editions. Because of its many notable qualities, this work stands preeminent in its special field.

[3] This is, of course, more an idea of *projective*, than of *elementary*, geometry.

drawn to cut the circles[1] in P, Q, P', Q', AP and BP', AQ and BQ' are pairs of parallel lines. There is similar obvious extension to spheres with tangent or secant lines.

In the first edition of his *Géomêtrie Descriptive*,[2] Monge derived the following results:

(A) *The six centers of similitude of three coplanar circles lie by threes on four straight lines.*

(B) *The vertices of the six common tangent cones of three spheres, taken in pairs, lie by threes on four straight lines.*

(C) *Given any four spheres in space fixed in magnitude and position, and the six cones tangent to them in pairs, externally; then the six vertices lie in a plane and indeed on four straight lines in the plane. If the six other tangent cones be drawn, then their vertices lie by threes in planes with threes of the first group.*

I believe that there has never been any question as to Monge's priority of discovery of theorem (C), about which I will add further comment presently. With regard to the particular case, theorem (A), (to which (B) is practically equivalent), a contemporary has been given some credit for its formulation. But, so far as I am aware, it has never before been suggested that it was known to the Greeks. In the following paragraphs I discuss these two views.

I.

Loria states:[3] "According to Fuss (*Nova Acta Petrop.*, T. XIV, 1805) Monge was inspired by d'Alembert." Kötter remarks:[4] "It has been noted by Grunert that Fuss published the Monge discussion [concerning circles], referring it back to d'Alembert," and a citation from *Nova Acta*, similar to the above, is given. Finally, as noticed by a correspondent of *L'Intermédiaire des Mathématiciens*,[5] according to Chasles,[6] Fuss attributes to d'Alembert the theorem for circles, while Carnot,[7] on the contrary, assigns it to Monge.

As Loria's statement is incorrect, while those of Grunert and Chasles are

[1] I consider the circle defined as a curve. It is to be noted, however, that this is *not* the definition generally used by modern writers. To be convinced of this it is sufficient to refer to the works of Enriques and Amaldi, Faifofer, Hadamard, and Ingrami.

[2] G. MONGE, *Géométrie descriptive. Leçons données aux écoles normales l'an de la republique* Paris . . . an VII [1798], pp. 54–55. In a new edition by Hachette, Paris, 1811, pp. 65–67. Hachette published in 1812, a supplementary volume, to pages 20–21 of which reference in this connection should be given. In English we have, *An elementary Treatise on descriptive geometry with a theory of shadows and of perspective: extracted from the French of G. Monge . . .* by J. F. Heather, London, 1851; our results are given on pp. 49–50. A free German translation of Monge's work, by G. Schreiber, was published at Karlsruhe and Freiburg, 1828. But a literal translation with notes, to which we will presently refer, was given in R. Haussner's edition, Ostwald's Klassiker, Nr. 117, Leipzig, 1900; our theorems occur on pages 68–70.

[3] In his history of "Perspektive und darstellende geomentrie," in CANTOR's *Vorlesungen über Geschichte der Mathematik*, Bd. 4 (1908), p. 629.

[4] E. KÖTTER, *Die Entwickelung der synthetischen Geometrie*, Leipzig, 1901, p. 112.

[5] Question 1418, 1898, p. 271 and 1911, p. 29. No reply has so far been published.

[6] CHASLES, *Aperçu historique*, 3e éd., Paris, 1889, p. 293.

[7] CARNOT, *Mémoire sur la relation qui existe entre les distances relatives à cinq points quelconques pris dans l'espace*, suivi d'un *Essai sur la théorie des transversales*, Paris, 1806, p. 87.

only partially correct, it may be well to set forth exactly what Fuss did give in this connection.

The paper of Fuss, referred to above, was presented to the Academy, July 4, 1799, and entitled, " Demonstrations de quelques théorèmes de géométrie." It commences as follows:[1] " Several years ago a young Frenchman, then employed in the ' Corp *Imperial* ' of the ' Cadets de Terre,' spoke to me about a geometrical theorem which, at the time when he was at Paris in the Royal military school, had some celebrity and was attributed to the late M. d'Alembert. Of this theorem I gave him a demonstration which I have recently found on looking over my papers." Fuss then takes up the theorem concerning the *external* centers of similitude of three circles (and this seems to be the theorem which he believed due to d'Alembert, although the more general one of Monge had been already published), before proceeding to properties "not less remarkable."

He considers four circles of different radii and in the same plane, and shows that the six *external* centers of similitude lie by threes on four straight lines. And, *for n coplanar circles, the* $n(n-1)/2!$ *external centers of similitude are situated, in general, by threes on* $n(n-1)(n-2)/3!$ *different straight lines.*

This is extended to n spheres with their centers in the same plane. The vertices of the cones tangent to the spheres in pairs (the centers of the spheres in each case on the same side of the vertex of the tangent cone) have exactly the properties indicated above for the centers of similitude.

Next, in taking up the corresponding theorems for a sphere, Fuss shows that: *If three small circles are enclosed in pairs by two great circles tangent to them the intersections of the three pairs of great circles are situated on the same great circle.* A similar theorem is given for n small circles on a sphere.

In none of the discussion are *internal* centers of similitude mentioned. On the other hand, as we have seen above, Monge stated three general theorems, namely (A), (B), (C), concerning both internal and external centers of similitude.

To make clear all that is implied in Theorem (C) it may be well to state the results symbolically.[2] If $E_{m,\,n}$ denote the external, and $I_{m,\,n}$ the internal centers of similitude of the spheres S_m and S_n ($m, n = 1, 2, 3, 4$ and $m \gtreqless n$), the following groups of points lie in planes:

$$\left.\begin{array}{l} E_{1,\,2},\ E_{1,\,3},\ E_{1,\,4},\ E_{2,\,3},\ E_{2,\,4},\ E_{3,\,4}; \\ E_{1,\,2},\ E_{1,\,3},\ I_{1,\,4}\ \ E_{2,\,3},\ I_{2,\,4},\ I_{3,\,4}; \\ E_{1,\,2},\ I_{1,\,3},\ E_{1,\,4},\ I_{2,\,3},\ E_{2,\,4},\ I_{3,\,4}; \\ I_{1,\,2},\ E_{1,\,3},\ E_{1,\,4},\ I_{2,\,3},\ I_{2,\,4},\ E_{3,\,4}; \\ I_{1,\,2},\ I_{1,\,3},\ I_{1,\,4},\ E_{2,\,3},\ E_{2,\,4},\ E_{3,\,4}. \end{array}\right\}$$

Monge overlooked the three planes:[3]

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[1] *Nova Acta Ac. Sc. imp. Petropolitanae*, Tome 14 (1797–1798), Petropoli, 1805, p. 139.

[2] *Cf.* HAUSSNER, *l. c.*, p. 200.

[3] LORIA makes two more errors (*l. c.*) by asserting that in theorem (C) "findet man einige Unrichtigkeiten, da Monge die Ebenen nicht betrachtete, von denen jede zwei innere und vier

$$\left.\begin{array}{l} I_{1,2},\ I_{1,3},\ E_{1,4},\ E_{2,3},\ I_{2,4},\ I_{3,4}; \\ I_{1,2},\ E_{1,3},\ I_{1,4},\ I_{2,3},\ E_{2,4},\ I_{3,4}; \\ E_{1,2},\ I_{1,3},\ I_{1,4},\ I_{2,3},\ I_{2,4},\ E_{3,4}. \end{array}\right\}$$

It is now apparent that so far as the testimony of Fuss[1] is concerned the name of d'Alembert should be associated with a very small portion, at most, of the theorem concerning the centers of similitude of three circles,[2] and that the terms " theorem of d'Alembert "[2] or " lines of d'Alembert,"[3] employed in connection with discussion of the four axes of similitude, are introduced both incorrectly and in a way to do injustice to discoveries of Monge.

II.

Let us now consider the question, Who was the discoverer of the centers of similitude[4] of two circles? Cantor asserts very definitely[5] that François Viète[6] [1540–1603] should be so considered. But from what follows it will be clear that many hundred years before Viète's time, Greeks were familiar with the points and several of their properties.

My argument is based almost wholly upon portions of the Mathematical Collections[7] of Pappus (*c.* 300 A. D.) and, more particularly, upon that part of it which describes the work *On Tangencies* by Apollonius of Perga (*c.* 225 B. C.). Among many other propositions which come up incidentally Pappus proves the following:

1. If two circles touch internally or externally and through the point of contact any two lines be drawn, the chords joining the points of intersection of these lines with each circle are parallel (pp. 832 f., 826 f.).

2. If two circles do not meet, the common direct tangent passes through the external center of similitude[8] (pp. 850 f.).

äussere Ähnlichkeitspunkte enthält." From the above it is clear that there are no " Unrichtigkeiten " in the theorem. Monge simply did not notice the three planes (here remarked) through four internal and two external centers of similitude (*not* two internal and four external as Loria asserts). In this connection Haussner also makes a slip (l. c., p. 199.)

[1] That is, in connection with the 1799 memoir cited above.

[2] I have vainly searched d'Alembert's *Opuscules mathématiques*, 8 tomes, Paris, 1761–1780, for any reference to this theorem.

[3] Used, for example, by F. G. M. *Exercices de Géométrie*, 5e éd., 1912, pp. 81, 146, 1260, 1272.

[4] The term *center of similitude* is due to Euler: " De centro similitudinis," *Nova Acta acad. sc. Petrop.*, Vol. 9 (1791), 1795, p. 154—" Conventuri exhib. die 23 Octob. 1777."

[5] M. CANTOR, *Vorlesungen über Geschichte der Mathematik*, Bd. 2_2 (1900), pp. 590–591.

[6] F. VIÈTE, *Apollonius Batavus*, Paris, 1600; reprinted in *Opera mathematica*, Lugd. Bat., 1646, pp. 325–338; also reprinted by J. W. CAMERER in *Apollonii de tactionibus quae supersunt*, Gothae et Amstelodami, 1795, pp. 1–58 at end.

[7] Pappi Alexandrini Collectionis . . . edidit . . . Hultsch. Berolini, I (1876), pp. 195–201; II (1877), pp. 644–649; 820–853.

[8] Aristarchus (310–250 B. C.) gave a more general result. It is Proposition 1 of his work " On the Sizes and Distances of the Sun and the Moon ": " Two equal spheres are comprehended by one and the same cylinder, and two unequal spheres by one and the same cone which has its vertex in the direction of the lesser sphere; and the straight lines drawn through the centers of the spheres is at right angles to each of the circles in which the surface of the cylinder, or of the cone,

3. If a circle C_3 is externally tangent to two other circles C_1, C_2, externally tangent to each other, the points of tangency of C_1C_3, C_2C_3, lie in a line with the external center of similitude of C_1C_2 (p. 208 f.).

4. The line joining the ends of two parallel radii drawn in opposite directions, of two equal circles, passes through the internal center of similitude (pp. 194 f.).

To sum up, we here find fundamental theorems concerning (*a*) internal and external centers of similitude of tangent circles; (*b*) the external center of similitude of unequal circles and (*c*) the internal center of similitude of equal circles. We have also, in 3, a particular case of part of Theorem (*A*) of Monge: *Given three circles C_1, C_2, and C_3, the internal centers of similitude of C_1C_3 and C_2C_3 are in a line with the external center of similitude of C_1 and C_2.*

I think further that it may be pretty conclusively shown, that the Greeks were familiar with the idea of the centers of similitude of two circles, in the general case and not alone in the particular cases referred to above. To make this clear I must recall a lemma given by Pappus in connection with his account of the second book of Apollonius *On Tangencies.* The main problem of this work is to describe a circle tangent to three given circles. If we regard lines and points as limiting cases of circles, we get ten types of problems, all of which were considered by Apollonius. In the second book only two types remain to be considered, namely:[1] (1) to describe a circle tangent to two given lines and to a given circle; (2) to describe a circle tangent to three given circles.

Now the famous lemma which Pappus gave for the solution of this latter problem is the following: "Given a circle in position, and the points D, E, F on a straight line; it is required to draw, to a point A of the circle, DA, AE to meet the circle again in B and C, such that BC lies in a line with CF."[2]

How can this lemma be used to solve the Problem of Apollonius? Pappus does not tell us. Viète and hundreds of others who have given solutions of the problem use methods in no wise involving the lemma; these methods can therefore bear little relation to that of Apollonius. I believe that Robert Simson, who did such signal service in the restoration of Euclid's *Porisms,* was the first to conjecture as to which one of the problems in *Tangencies* the lemma was subsidiary. He has written: "Often indeed have I revolved the subject in my mind, but I have never succeeded in arriving at any satisfactory conclusion; except that the lemma, by no uncertain marks, appeared to be necessary for the following problem: two circles and a point being given by position, it is required to describe a third circle which shall touch the given circles and pass through the given point.

touches the spheres." *Cf. Aristarchus of Samos the Ancient Copernicus,* ed. by T. L. Heath, Oxford, 1913, pp. 354–355. In the course of the proof of this proposition it is shown that the vertex of the cone is a center of similitude.

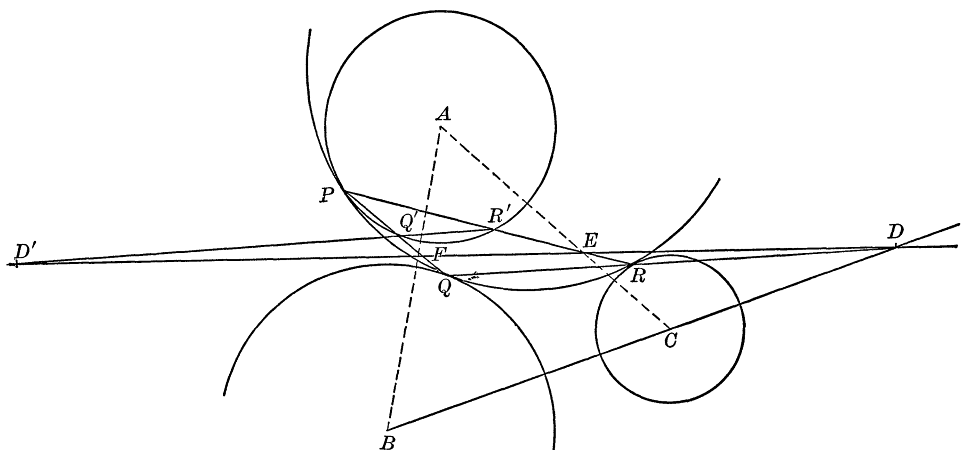
[1] Pappus, *l. c.*, pp. 646–7.

[2] The generalization of this lemma to the case where D, E, F, are not necessarily collinear, was first found by Robert Simson in 1731. The further generalization for any polygon inscribed in a circle was given by Giordano di Ottajano, a youth of sixteen, in 1784. Early in the nineteenth century, with the birth of the principle of duality and the development of projective geometry, the problem was solved for polygons circumscribed and inscribed to cones. Researches of Townsend, Potts and Renshaw led up to discoveries of Sir William Rowan Hamilton regarding polygons inscribed in a sphere, or an ellipsoid, or an hyperboloid, and with sides passing through given points.

In what manner, however, the lemma might be subsidiary to this problem I did by no means perceive. I have directed my attention to the solutions of Viète and others, hoping that by chance I might hit upon the analysis requiring this lemma; but in vain until this day, after various trials, I discovered the true analysis of Apollonius,—to which indeed both this Prop. 117 of Pappus as well as Props. 116 and 118 are manifestly subsidiary. February 9, 1734."[1] Simson then proceeds to set forth the analysis and synthesis of his solution after the Greek manner.

From editions of Pappus in his day, Simson could not learn that while the problem he attacked was in the first, the lemma he referred to was in the second, book of the *Tangencies*. With very similar reasoning, however, his method might have been extended to the case of a circle tangent to three given circles. Doubtless independent of Simson, this was first done, I believe, by Scorza in 1819.[2] In effect his method was the following.

Let A, B, C be the centers of the three given unequal circles which are also named by these letters. Let a, b, c be their respective radii. Moreover, suppose that the circles do not meet and that no one is inside another; then there are eight solutions corresponding to the different cases. It will suffice to consider a single case, when, say, the circle PQR is tangent to A internally at P, and to B, C externally, at Q, R respectively. Then QR passes through D the external center of similitude of B and C (Pappus, p. 210); and PR, PQ pass through E, F the internal centers of similitude of A and C, A and B respectively.



Let PR, PQ meet the circle A again in R' and Q', and let $R'Q'$ meet DF in D'. Then since QR is parallel to $Q'R'$ (Pappus, p. 832), D' is fixed by the ratio $DF/D'F = b/a$.

[1] R. Simson, *Opera Quaedem Reliqua*, Glasquae, MDCCLXXVI Appendix, pp. 20–23. *Cf.* T. S. Davies in *The Mathematician*, March, 1848, Vol. 3, p. 78.

[2] G. Scorza in " Divinazione della soluzione Apolloniana del problema de tre cerchi " by V. Flauti, *Atti della Reale Accademia delle Scienze e Belle Lettere*, Vol. 1 (1819), 77–78.

Hence, if E lies in the line DF,[1] the solution of the case of the problem we are considering is reduced to inscribing in the circle A a triangle $PQ'R'$, whose side produced shall pass through the fixed collinear points E, F, D'. This can be done by the lemma. The other cases may be treated in a similar way.

In the above I have neither paused to elaborate the details of the various steps nor endeavored to set forth the whole in the Greek manner. To fill in these lacunae, the interested reader may turn to the writings of Pappus, Simson, Flauti, Scorza, and Zeuthen[2]—for the weighty testimony of Zeuthen favors the solution just given as the most probable original of Apollonius.

In the course of this restoration not only have various properties of centers of similitude of circles, mentioned earlier in this paper, been used, but Monge's Theorem (A) has also been necessary. I therefore hold that Theorem (A) and centers of similitude were discovered by the Greeks. How much the more is the assertion concerning "Monge's Theorem" confirmed, when we recall that it follows immediately, on applying to the triangle ABC, the converse of the theorem of Menelaus of Alexandria (c. 80 A. D.) with regard to transversals.[3]

A CARDIOIDOGRAPH.

By C. M. HEBBERT, University of Illinois.

In his book called "Linkages"[4] J. D. C. DeRoos illustrates a rather complicated linkage for describing a cardioid. A simpler form of cardioidograph is presented in this paper.

The cardioid may be defined as the path traversed by any point of the circumference of a circle as it rolls upon a fixed coplanar circle of the same radius. In figure 1 consider the cardioid which is the path of the point B of the rolling

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[1] Recalling the theorem of Aristarchus referred to above (note 8, page 9) a possible Greek mode of proof (followed by Monge) that D, E, F are collinear, is the following. Consider A, B, C as great circles of three spheres. If the two transverse planes tangent to the sphere A and to the spheres B and C, be drawn, these planes will each pass through the three centers of similitude, D, E, F; that is, these points lie on the line of intersection of the tangent planes. A second method of proof, in the Greek manner, is indicated below in note 3.

[2] H. G. ZEUTHEN, *Die Lehre von den Kegelschnitten im Altertum*, Deutsche Ausgabe besorgt von R. v. Fischer-Benson. Kopenhagen, 1886, pp. 381–383.

[3] In the above figure, for instance,

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{a}{b}\cdot-\frac{b}{c}\cdot\frac{c}{a}=-1,$$

and conversely. This is Lemma I, Book III, of the *Spherics* of Menelaus; or, to be more accurate, the lemma was stated in the form: If DEF is a transversal of the triangle ABC, the ratio of AF to FB is equal to the ratio compounded of the ratios CD to BD and CE to EA. (*Theodosii sphaericorum Lib. III* . . . *Menelai Sphaericorum Lib. III* . . . Messanae, 1558, pp. 36 *verso*—37 *recto* of second pagination, or p. 83 of *Menelai sphaericorum Libri III* . . . *curavit vir Cl. Ed. Halleius*. Oxonii, MDCCLVIII.)

[4] D. Van Nostrand, New York, 1879. English translation from the *Revue Universelle des Mines*.